G-2/230/21 Roll No. ....

M.Sc. II Semester Examination, 2021

MATHEMATICS

Paper I

(Advanced Abstract Algebra-II)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections. Section A is objective type/multiple choice questions with no internal choice. Section B is short answer type with internal choice. Section C is long answer type with internal choice.*

SECTION A **1×10=10**

(Objective Type Questions)

Choose the correct answer :

1. Which of the following statements is true :

(a) Every artinian ring is noetherian

(b) Every artinian module is noetherian

(c) In a noetherian ring every prime ideal is maximal

(d) Every noetherian ring is artinian

2. If f be a homomorphism of a R-module M into R-module N, then f will be isomorphism if :

(a) $\ker f = \{0\}$ (b) $\text{Img } f = \{0\}$

(c) $\text{Img } f \neq (0)$ (d) $\ker f \neq (0)$

P.T.O.

3. A linear transformation $T \in A_F(v)$ for which $T^n = 0$ for some positive n is called :

(a) Nilpotent (b) Cyclic

(c) Both (a) and (b) (d) None is correct

4. The characteristic and minimal polynomial for a linear operator T have :

(a) Same roots (b) Different roots

(c) Sometimes different and sometimes same

(d) None of these

5. Matrix A is Nilpotent of index K, then A^n, $n > 1$ is also Nilpotent of index :

(a) $k - 1$ (b) $k + 1$

(c) index $\leq k$ (d) None of these

6. Two Nilpotent linear transformation are similar if and only if they have :

(a) Same invariant

(b) Different invariant

(c) Non-equivalence relation

(d) Same-Antisymmetric Relation

7. If $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{k-1} x^{k-1} + x^k$ be a monic polynomial over a field F, then companian matrix of $f(x)$ is :

(a) $1 \times k$ matrix (b) $k \times 1$ matrix

(c) $k \times k$ matrix (d) None of the above

8. Let A be an $m \times n$ matrix over a principal ideal domain, then which of the following is correct :

(a) Row rank A = Column rank A

(b) Row rank A < Column rank A

(c) Row rank A > Column rank A

(d) None of these

9. Let $T \in \text{Hom}_F(V, V)$ if V as an $F[x]$ module relative to T is cyclic, then V is called :

(a) T-cyclic space (b) T-simple space

(c) T-subspace (d) T-invariant subspace

10. Rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$ is :

(a) 1 (b) 2

(c) 3 (d) None of these

SECTION B **4 × 5 = 20**

(Short Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. If M is an R module and $x \in M$, then prove that the set $R_x = \{r_x : r \in R\}$ is an R-submodule of M.

Or

Prove that quotient module and Naetherian module is Noetherian.

Unit-II

2. Prove that the minimal polynomial of a linear operator $T \in A(V)$ divides its characteristic polynomial.

Or

Let A be an algebra, with unit element over F and let dimension of A over F be m, then prove every element in A satisfies some non-trivial polynomial in $F[x]$ of degree at most m.

Unit-III

3. Prove that a Jordan block J may be written as the sum of a scalar matrix and a Nilpotent.

Or

Prove that the matrix $A = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$ is Nilpotent and find its index.

Unit-IV

4. If R is a PID and $a \in R$ is a unit, then prove that $\dfrac{R}{Ra} = (0)$.

Or

Show that the abelian group generated by x_1 and x_2 subject to $2x_1 = 0$, $3x_2 = 0$ is isomorphic to $\frac{Z_1}{(6)}$.

Unit-V

5. Find out the rational canonical form of the matrix whose invariant factors are $(x-3)$, $(x-3)(x-1)$, $(x-3)(x-1)^2$.

Or

Reduce the matrix $A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$ into rational canonical form.

SECTION C **10 × 5 = 50**

(Long Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. Define free module and show that if M be a free R module with a basis $\{e_1, e_2, \ldots, e_n\}$, then $M \cong R^n$.

Or

Let R be a ring with unity. Show that an R

module M is cyclic if and only if $M \cong \frac{R}{I}$ for some left ideal I of R.

Unit-II

2. Let $\lambda \in F$ be a characteristic root of $T \in A(V)$. Prove that for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

Or

Let the linear transformation $T \in A_F(V)$ be Nilpotent, then prove that $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_m T^m$ where $\alpha_i \in F$, $0 \le i \le m$ is invertible if $\alpha_0 \neq 0$.

Unit-III

3. If $u \in V_1$ is such that $uT^{n_1-k} = 0$, where $0 < k < n$, then prove that $u = u_0 T^k$ for some $u_0 \in V_1$.

Or

Find the Jordan Canonical form of $\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$.

Unit-IV

4. Let A be an $(m \times n)$ matrix over R if

$$E_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$$

then prove that $E_{ij}A$ is the matrix obtained from A be interchanging the i^{th} and j^{th} rows and $E_{ij}^{-1} = E_{ij}$.

Or

Find the rank of the subgroup of z^4 generated by the following list of elements (2, 3, 1, 4), (1, 2, 3, 0), (1, 1, 1, 4).

Unit-V

5. Let M be a finitely generated module over a principal ideal domain R, then prove that

$$M = F \oplus \text{Tor M}$$

where (i) $F \simeq R^S$ for some non-negative integers,

(ii) $\text{Tor M} = \frac{R}{Ra_1} \oplus \frac{R}{Ra_2} \oplus ..... \oplus \frac{R}{Ra_r}$ where a_i are non-zero non-unit element in R such that $| a_1 | a_2 | a_3 ..... | a_r$.

Or

Find the rational canonical form of the matrix A over ϕ, where

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 8 & 3 \end{bmatrix}.$$

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